

MINIMAL GROMOV–WITTEN RING

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Abstract. We build the abstract theory of Gromov–Witten invariants of genus 0 for quantum minimal Fano varieties (a minimal natural (with respect to Gromov–Witten theory) class of varieties). In particular, we consider “the minimal Gromov–Witten ring”, i. e. a commutative algebra with generators and relations of the form used in the Gromov–Witten theory of Fano variety (of unspecified dimension). Gromov–Witten theory of any quantum minimal variety is a homomorphism of this ring to \mathbb{C} . We prove the Abstract Reconstruction Theorem which states the particular isomorphism of this ring with a free commutative ring generated by “prime two-pointed invariants”. We also find the solutions of the differential equations of type DN for a Fano variety of dimension N in terms of generating series of one-pointed Gromov–Witten invariants.

Consider a smooth Fano variety V of dimension N . Let $H_H^*(V, \mathbb{Q}) \subset H^*(V, \mathbb{Q})$ be a subspace multiplicatively generated by the anticanonical class $H \in H^2(V, \mathbb{Q})$. It is tautologically closed with respect to the multiplication in cohomology. The Gromov–Witten theory (more precisely, the set of Gromov–Witten invariants of genus 0) enables one to “deform” the cohomology ring, that is, to define quantum multiplication in the graded space $QH^*(V) = H^*(V, \mathbb{Q}) \otimes \mathbb{C}[q]$. This multiplication coincides with the multiplication in $H^*(V, \mathbb{Q})$ under specification $q = 0$. The subspace $QH_H^*(V) = H_H^*(V) \otimes \mathbb{C}[q]$ is not closed with respect to the quantum multiplication in general. The variety is called quantum minimal if the corresponding subspace is closed.

Let V be quantum minimal. Then there is a natural submodule corresponding to $QH_H^*(V)$ in its quantum \mathcal{D} -module. The connection in it is given by a matrix of quantum multiplication by the anticanonical class of V (in the other words, by prime two-pointed genus zero Gromov–Witten invariants). After regularizing it one get the determinantal operator (the operator of type DN) for V . Such operators are deeply studied in [GS05]. One of the mirror symmetry conjectures states that this operator coincides with the Picard–Fuchs operator for dual to V Landau–Ginzburg model (see, for instance [KM94]). This is proved, for example, for threefolds, see [Pr04] and [Pr05].

In the paper we prove the following theorem. Let $\langle \tau_i H^j \rangle_d$ be the one-pointed Gromov–Witten invariant for the curve of the anticanonical degree d . Consider a regularized I -series (the generating series for one-pointed Gromov–Witten invariants)

$$\tilde{I}^V = 1 + \sum_{0 \leq j \leq N, d > 0} \langle \tau_{d+j+2} H^{N-j} \rangle_d q^d h^j / H^N \cdot (h+1) \cdot \dots \cdot (h+d) \in \mathbb{C}[[q]][h]/h^N.$$

Theorem (Corollary 2.2.6). *Let $\tilde{I}^V = \sum_{0 \leq k \leq N} \tilde{I}^k h^k$, where \tilde{I}^i for any i is the series on q . Then N functions*

$$\tilde{I}^0, \quad \tilde{I}^0 \log(q) + \tilde{I}^1, \quad \tilde{I}^0 \log(q)^2 / 2! + \tilde{I}^1 \log(q) + \tilde{I}^2, \quad \dots$$

form the basis of the kernel of the operator of type DN for V .

The idea of the expression of the solutions of equations of type DN in terms of Gromov–Witten invariants is going back to Witten, Dijkgraaf, and Dubrovin. The proof of this theorem is based on the relations between Gromov–Witten invariants of genus 0. This enables us to generalize Gromov–Witten theories of quantum minimal Fanos and their determinantal operators to the formal theory which does not depend on the variety, and, moreover, on its dimension. Numerical invariants in this theory become the universal polynomials that are unique for all quantum minimal Fanos of any dimension.

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Fix $N \in \mathbb{N}$. Let $A_N = \mathbb{C}[a_{ij}]$, $0 \leq i, j \leq N$. Let $\mathcal{D} = \mathbb{C}[q, \frac{d}{dq}]$. Put $D = q \frac{d}{dq}$. Consider the matrix

$$M = \begin{pmatrix} a_{00}(Dq) & a_{01}(Dq)^2 & \dots & a_{0,N-1}(Dq)^N & a_{0,N}(Dq)^{N+1} \\ 1 & a_{11}(Dq) & \dots & a_{1,N-1}(Dq)^{N-1} & a_{1,N}(Dq)^N \\ & & \dots & & \\ 0 & 0 & \dots & 1 & a_{NN}(Dq) \end{pmatrix}$$

with entries in $A_N \otimes \mathcal{D}$. Define the operator $L_N \in A_N \otimes \mathcal{D}$ by

$$\det_{\text{right}}(D - M) = DL_N^1$$

(as $A_N \otimes \mathcal{D}$ is non-commutative, the determinant is taken with respect to the rightmost column).

Example. Consider operators L_N as (non-commutative) polynomials in q and D . It is easy to check that the degrees of such operator in q and D do not depend on its representation as a polynomial. As a polynomial, L_2 (resp. L_3) is of degree 2 (resp. 3) in D and of degree 3 (resp. 4) in q . Write down the analytic solutions of $L_2\Phi = 0$ and $L_3\Phi = 0$ in terms of a_{ij} 's.

$$\begin{aligned} &1 + a_{00}q + (1/2a_{01} + a_{00}^2)q^2 + (7/6a_{01}a_{00} + a_{00}^3 + 1/3a_{01}a_{11} + 2/9a_{02})q^3 + \\ &(23/12a_{01}a_{00}^2 + 1/4a_{01}a_{11}^2 + 3/8a_{01}^2 + 1/8a_{22}a_{02} + 5/6a_{11}a_{00}a_{01} + 3/16a_{01}a_{12} + \\ &43/72a_{02}a_{00} + a_{00}^4 + 1/6a_{02}a_{11})q^4 + O(q^5). \end{aligned} \quad (L_2)$$

$$\begin{aligned} &1 + a_{00}q + (1/2a_{01} + a_{00}^2)q^2 + (7/6a_{01}a_{00} + a_{00}^3 + 1/3a_{01}a_{11} + 2/9a_{02})q^3 + \\ &(23/12a_{01}a_{00}^2 + 1/4a_{01}a_{11}^2 + 3/8a_{01}^2 + 1/8a_{22}a_{02} + 5/6a_{11}a_{00}a_{01} + 3/16a_{01}a_{12} + \\ &43/72a_{02}a_{00} + a_{00}^4 + 1/6a_{02}a_{11} + 3/32a_{03})q^4 + O(q^5). \end{aligned} \quad (L_3)$$

The first few terms of the solutions coincide. Moreover, the solutions are “the same” if a_{ij} 's that are “out of bounds” vanish, i. e. if we put $a_{i3} = 0$ in the solution of $L_3\Phi = 0$, we get the solution of $L_2\Phi = 0$.

This same turns out to be true in the general case. Fix two natural numbers $N_1 < N_2$. Let Φ_1 and Φ_2 be the analytic solutions of L_{N_1} and L_{N_2} , normalized by $\Phi_i(0) = 1$. Let Φ'_2 be the series given by putting $a_{ij} = 0$ for $N_1 < i, j \leq N_2$ in Φ_2 . Then $\Phi_1 = \Phi'_2$. Moreover, it is not difficult to prove (see Lemma 1.1.3 and Proposition 2.2.2 below) that for any n if $N_1 \gg n$, then

$$\Phi_1 \mod q^n = \Phi_2 \mod q^n.$$

A similar statement holds for the logarithmic solutions of $L_N\Phi = 0$'s.

So, to get the solutions of such equation, one should “restrict” the solutions of the equation of larger index. Moreover, the first few terms of analitic expansions of different solutions coincide. Thus, we can define “the universal series”. “The restrictions” of this series to A_N are the solutions of equations $L_N\Phi = 0$. This series is “the generating series of abstract one-pointed Gromov–Witten invariants” in the following sense.

Below we define *the minimal Gromov–Witten ring* GW as a commutative algebra, with generators and relations of the form used in the Gromov–Witten theory. Our definition is similar to Dubrovin's definition of formal Frobenius manifold or Kontsevich–Manin's treatment to the theory of Gromov–Witten invariants. The difference is that we do not fix the dimension and consider “the abstract Gromov–Witten invariants of a Fano variety of unspecified dimension”. For this we consider “invariants for the classes with one class replaced by the Poincaré dual one” and reformulate Kontsevich–Manin axioms in terms of such “invariants”.

Definition of the minimal Gromov–Witten ring. Consider formal symbols of the form

$$\langle \tau_{d_1} H^{i_1}, \dots, \tau_{d_{n-1}} H^{i_{n-1}}, \tau_{d_n} H_r \rangle,$$

$n \geq 1$, $i_1, \dots, i_{n-1}, r, d_1, \dots, d_n \in \mathbb{Z}_{\geq 0}$ (the last term is indexed by a subscript!). We write H^i , H_j instead of $\tau_0 H^i$, $\tau_0 H_j$ for simplicity. Define *the degree* of such symbol as a number $\sum d_s + \sum i_s - r + (3 - n)$.

¹The operators that come from a symmetric with respect to the anti-diagonal matrices (i. e. ones with $a_{ij} = a_{N-j, N-i}$) are called the operators of type DN .

Let F be the set of all symbols with non-negative degrees. *The minimal Gromov–Witten ring* is a graded ring

$$GW = \mathbb{C}[F]/\text{Rel},$$

where Rel is the ideal generated by the following relations.

GW1 (S_n -covariance axiom, cf. [KM94], 2.2.1): Consider any permutation $\sigma \in S_{n-1}$. Let $j_k = i_{\sigma(k)}$ and $f_k = d_{\sigma(k)}$. Then

$$\langle \tau_{d_1} H^{i_1}, \dots, \tau_{d_{n-1}} H^{i_{n-1}}, \tau_{d_n} H_r \rangle = \langle \tau_{f_1} H^{j_1}, \dots, \tau_{f_{n-1}} H^{j_{n-1}}, \tau_{d_n} H_r \rangle.$$

GW2 (normalization, cf. [KM98], 1.4.1): Let $r = \sum d_s + \sum i_s + (3 - n)$. Then

$$\langle \tau_{d_1} H^{i_1}, \dots, \tau_{d_{n-1}} H^{i_{n-1}}, \tau_{d_n} H_r \rangle = \frac{(d_1 + \dots + d_n)!}{d_1! \dots d_n!} \cdot M,$$

where $M = 1$ if $\sum d_j = n - 3$ and $M = 0$ otherwise.

GW3 (fundamental class axiom or string equation, cf. [Ma99], VI–5.1):

$$\langle H^0, \tau_{d_1} H^{i_1}, \dots, \tau_{d_{n-1}} H^{i_{n-1}}, \tau_{d_n} H_r \rangle = \sum_{j=1}^n \langle \tau_{d_1} H^{i_1} \dots \tau_{d_{i-1}} H^{i_{j-1}}, \tau_{d_{i-1}} H_{i_j}, \tau_{d_{i+1}} H^{i_{j+1}}, \dots, \tau_{d_n} H_r \rangle,$$

except for the case $\langle H^0, H^i, H_i \rangle$, which is given by GW2.

GW4 (divisor axiom, cf. [Ma99], VI–5.4):

$$\begin{aligned} \langle H^1, \tau_{d_1} H^{i_1}, \dots, \tau_{d_{n-1}} H^{i_{n-1}}, \tau_{d_n} H_r \rangle &= d \cdot \langle \tau_{d_1} H^{i_1}, \dots, \tau_{d_n} H_r \rangle + \\ &\sum_{s=1}^{n-1} \langle \tau_{d_1} H^{i_1}, \dots, \tau_{d_{s-1}} H^{i_{s-1}}, \dots, \tau_{d_n} H_r \rangle + \langle \tau_{d_1} H^{i_1}, \dots, \tau_{d_{n-1}} H_{r-1} \rangle. \end{aligned}$$

where $d > 0$ is the degree of the left side.

GW5 (topological recursion, cf. [Pa98], formula 6): For any numbers $c_1, \dots, c_n, i_1, \dots, i_n$ and set $S \subset \{1, \dots, n\}$ denote the sequence $\tau_{c_{s_1}} H^{i_{s_1}}, \dots, \tau_{c_{s_k}} H^{i_{s_k}}$ (s_1, \dots, s_k are different elements of S) by \coprod_S . For any $n \geq 0$

$$\langle \coprod_{\{1, \dots, n\}}, \tau_{d_1} H^{j_1}, \tau_{d_2} H^{j_2}, \tau_{d_3} H_r \rangle = \sum \langle \tau_{d_1-1} H^{j_1}, \coprod_{S_1}, H_a \rangle \langle H^a, \coprod_{S_2}, \tau_{d_2} H^{j_2}, \tau_{d_3} H_r \rangle$$

and

$$\langle \coprod_{\{1, \dots, n\}}, \tau_{d_1} H^{j_1}, \tau_{d_2} H^{j_2}, \tau_{d_3} H_r \rangle = \sum \langle \coprod_{S_1}, \tau_{d_1} H^{j_1}, \tau_{d_2} H^{j_2}, H_a \rangle \langle H^a, \coprod_{S_2}, \tau_{d_3-1} H_r \rangle,$$

where the sums are taken over all splittings $S_1 \sqcup S_2 = \{1, \dots, n\}$ and all $a \in \mathbb{Z}_{\geq 0}$ such that the degrees of the symbols in the expression are non-negative (notice that the sum is finite).

It turns out that GW has a convenient multiplicative basis. Consider a ring $A = \mathbb{C}[a_{ij}]$, $0 \leq i \leq j$, $j > 0$ and the map $r: A \rightarrow GW$ given by $a_{ij} \rightarrow \langle H, H^j, H_i \rangle$.

Theorem 3.1 (the Abstract Reconstruction Theorem). *The map r is an isomorphism.*

This theorem is an abstract version of the First Reconstruction Theorem of Kontsevich and Manin ([KM94], Theorem 3.1).

Let $a_{00} = 0$ (“geometric case”, see Remark 3.5 for the general case). Via r one may view the coefficients of $L_N \Phi = 0$ ’s and of their solutions as lying in GW .

Consider the series

$$\tilde{I} = 1 + \sum_{j \geq 0, i > j-2} \langle \tau_i H_j \rangle q^{i-j+2} h^j (h+1) \dots (h+i-j+2) \in A \otimes \mathbb{C}[[q]][[h]].$$

In terms of a_{ij} ’s it rewrites as

$$\begin{aligned} \tilde{I} = 1 + (a_{11}h + (a_{22} - a_{11})h^2)q + &\left(\frac{a_{01}}{2} + \left(\frac{a_{01}}{4} + \frac{a_{11}^2}{2} + \frac{a_{12}}{4} \right)h + \right. \\ &\left. \left(-\frac{a_{01}}{8} + \frac{a_{23}}{8} + \frac{a_{11}a_{22}}{2} - \frac{a_{11}^2}{4} + \frac{a_{22}^2}{4} \right)h^2 \right)q^2 + O(q^3, h^3). \end{aligned}$$

This universal generating series of one-pointed Gromov–Witten invariants defines the solutions of $L_N\Phi = 0$ for all N . Namely, Theorem 3.1 and Corollary 2.2.6 directly imply the following theorem. For any \mathbb{C} -algebra R let $r_N: A \otimes R \rightarrow A_N \otimes R$ be the map given by $a_{ij} \mapsto a_{ij}$ if $0 \leq i, j \leq N$ and $a_{ij} \mapsto 0$ otherwise. Define $\tilde{I}^s \in A \otimes \mathbb{C}[[q]]$ by $\tilde{I} = \sum \tilde{I}^s h^s$. Define S_i 's by $S_0 = \tilde{I}^0$, $S_1 = \tilde{I}^0 \log(q) + \tilde{I}^1$, $S_2 = \tilde{I}^0 \log(q)^2/2! + \tilde{I}^1 \log(q) + \tilde{I}^2$ and so on.

Theorem 3.3. *The set $\{r_N(S_0), \dots, r_N(S_{N-1})\}$ is the basis for the space of solutions of the differential equation $L_N\Phi = 0$.*

Corollary 2.2.6 is the particular case of Theorem 3.3. The proofs of these two results are almost identical. According to this reason in the first part of the paper (sections 1 and 2) we prove (without using of the abstract Gromov–Witten theory) Corollary 2.2.6 that have direct applications in the studying of the geometry of Fano varieties. In the second part (Section 3) we prove Theorem 3.1. Using this theorem we formally conclude Theorem 3.3 from Corollary 2.2.6.

The paper is organized as follows. In Section 1 we consider a quantum minimal Fano variety V of dimension N . The two-pointed Gromov–Witten invariants give the quantum connection and the differential operator associated with it. The elements of kernel of this quantum differential operator are given by the I -series of V (i. e. the generating series of one-pointed Gromov–Witten invariants of V). In Section 2 we study regularization of the quantum differential operator, which gives the operator of type DN . The Frobenius method gives the explicit expressions for the solutions of DN associated with V in terms of its I -series. All proofs in Section 1 and Section 2 are based on the fundamental class axiom, the divisor axiom, and the topological recursion relations for V . In Section 3 we prove the Abstract Reconstruction Theorem and the abstract version of theorems of Section 1 and Section 2, using the same arguments in the abstract setup. We give the explicit recursive relations for all solutions of the differential operators in the Appendix.

1. QUANTUM OPERATORS

1.1. Non-commutative determinants. Let R be an associative \mathbb{C} -algebra (not necessary commutative). We consider matrices with entries in R . The indices of matrix M of size $N+1$ run from 0 to N . The submatrix of size $i \times i$ that is the NW corner of M is called *the i -th leading principal submatrix*.

1.1.1. Definition [[GS05], Definition 1.3]. The matrix M with elements in R is called *almost triangular* if $M_{ij} = 0$ for $i+1 > j$ and $M_{i+1,i} = -1$.

1.1.2. Definition [[GS05], Definition 1.2]. Consider the matrix M with elements in R . The *right determinant* of M is the determinant taken with respect to the rightmost column:

$$\det_{\text{right}}(M) = \sum_{i=0}^N M_{iN} C_{iN},$$

where C_{iN} are cofactors taken as right determinants.

For any $(N+1) \times (N+1)$ -matrix $M = (M_{ij})_{0 \leq i,j \leq N}$ define the matrix M^τ by $M_{ij}^\tau = M_{N-j, N-i}$ (i. e. M^τ is “transpose to M with respect to the anti-diagonal”).

1.1.3. Lemma [Golyshev, Stienstra, see [GS05], 1.4]. *Let M be an almost triangular $(N+1) \times (N+1)$ -matrix. Put*

$$P_0 = 1, \quad P_{i+1} = \sum_{j=0}^i M_{ji} P_j.$$

Then P_i is the right determinant of i -th leading principal submatrix of M . In particular, $P_{N+1} = \det_{\text{right}}(M)$.

Proof. By induction on the size of the submatrices. For $i = 1$ this is trivial. Denote the $(i+1)$ -th leading principal submatrix of M by M_{i+1} . Notice that the right determinant of the matrix M_{i+1}^j obtained by deleting last column and j -th row from M_{i+1} equals

$$\det_{\text{right}} M_{i+1}^j = (-1)^{i-j} P_j.$$

Thus, we have

$$\det_{\text{right}} M_{i+1} = \sum_{j=0}^i (-1)^{j+i} M_{ji} \det_{\text{right}} M_{i+1}^j = \sum_{j=0}^i M_{ji} P_j = P_{i+1}.$$

□

1.1.4. Lemma [cf. Golyshev, Stienstra, Proposition 1.7 in [GS05]]. *Let M be an almost triangular matrix and let $\xi = (\xi_0, \dots, \xi_N)^T$. Let $M\xi = 0$. Then*

$$\det_{\text{right}} (M^T) \xi_N = 0.$$

Proof. We have the following system of equations on $\{\xi_i\}$:

$$\begin{cases} M_{0,0}\xi_0 + \dots + M_{0,N}\xi_N = 0 \\ -\xi_0 + M_{1,1}\xi_1 + \dots + M_{1,N}\xi_N = 0 \\ \dots \\ -\xi_{i-1} + M_{i,i}\xi_i + \dots + M_{i,N}\xi_N = 0 \\ \dots \\ -\xi_{N-1} + M_{N,N}\xi_N = 0. \end{cases}$$

Let P_i be given by Lemma 1.1.3 applied to the matrix M^T . Let us solve the system moving, step by step, in the reverse direction. We have $P_i \xi_N = \xi_{N-i}$. Thus,

$$\det_{\text{right}} (M^T) \xi_N = \left(\sum_{i=0}^N M_{iN}^T P_i \right) \xi_N = \sum_{i=0}^N M_{0i} \xi_i = 0.$$

□

1.2. Quantum operators. Consider a smooth Fano variety V of dimension N with $\text{Pic}(V) \cong \mathbb{Z}$. Denote $H = -K_V$ and $H^*(V) = H^*(V, \mathbb{Q})$. (The natural map $\text{Pic}(V) \rightarrow H^2(V, \mathbb{Z})$ is an isomorphism for smooth Fanos, so we use the same notation for the element of $\text{Pic}(V) \otimes \mathbb{Q}$ and for its class in $H^2(V)$.) Let $H_H^*(V) \subset H^*(V)$ be a divisorial subspace, that is one generated by the powers of H .

Let $\gamma_1, \dots, \gamma_n \in H^*(V)$ and d be the anticanonical degree of an effective algebraic curve $\beta \in H_2(V)$. We denote the respective Gromov–Witten invariant (of genus zero) with descendants of degrees $d_1, \dots, d_n \in \mathbb{Z}_{\geq 0}$ (see [Ma99], VI–2.1) by $\langle \tau_{d_1} \gamma_1, \dots, \tau_{d_n} \gamma_n \rangle_d$.

1.2.1. The subspace $H_H^*(V) \subset H^*(V)$ is tautologically closed with respect to the multiplication, i. e. for any $\gamma_1, \gamma_2 \in H_H^*(V)$ the product $\gamma_1 \cdot \gamma_2$ lies in $H_H^*(V)$. The multiplication structure on the cohomology ring may be deformed. That is, one can consider a *quantum cohomology ring* $QH^*(V) = H^*(V) \otimes \mathbb{C}[q]$ (see [Ma99], Definition 0.0.2) with quantum multiplication, $\star: QH^*(V) \times QH^*(V) \rightarrow QH^*(V)$, i. e. the bilinear map given by

$$\gamma_1 \star \gamma_2 = \sum_{\gamma, d} q^d \langle \gamma_1, \gamma_2, \gamma^\vee \rangle_d \gamma$$

for all $\gamma_1, \gamma_2, \gamma \in H^*(V)$, where γ^\vee is the Poincaré dual class to γ (we identify elements of $\gamma \in H^*(V)$ and $\gamma \otimes 1 \in QH^*(V)$). The constant term of $\gamma_1 \star \gamma_2$ (with respect to q) is $\gamma_1 \cdot \gamma_2$. The subspace $QH_H^*(V) = H_H^*(V) \otimes \mathbb{C}[q]$ is not closed with respect to \star in general. The examples of varieties V with non-closed subspaces $H_H^*(V)$ are Grassmannians $G(k, n)$, $k, n - k > 1$ of dimension > 4 (for instance, $G(2, 5)$) or their hyperplane sections of dimension ≥ 4 .

1.2.2. Definition. The variety V is called *quantum minimal* if $QH_H^*(V)$ is quantum closed, i. e. if for any $\gamma_1, \gamma_2 \in H_H^*(V)$, $\mu \in H_H^*(V)^\perp$ the Gromov–Witten invariant $\langle \gamma_1, \gamma_2, \mu \rangle_d$ vanishes².

In other words, the variety is quantum minimal if and only if $QH_H^*(V)$ is the subring of $QH^*(V)$.

Throughout the paper we assume V to be quantum minimal.

1.2.3. Consider a ring $B = \mathbb{C}[q, q^{-1}]$. Consider the basis $\{H^i\}$, $i = 0, \dots, N$, of $H_H^*(V)$ (where H^0 is the ring unity and $H = H^1$). Let H_i be the Poincaré dual for H^i . Consider the (trivial) vector bundle HQ

² A Fano variety is called minimal if its cohomology is as small as it can be (just \mathbb{Z} 's in every even dimension). Quantum minimal variety has as small “quantum anticanonical part” as it can be, that is, similar to the quantum cohomology of minimal one. That’s why it is natural quantum analog of classical minimal variety.

over $\text{Spec}(B)$ with fibers $H_H^*(V)$. Put $h^i = H^i \otimes 1 \in H_H^*(V) \otimes B$. Put $h = h^1$ and $k_V = K_V \otimes 1 = -h$. Let $S = H^0(HQ)$. As $S \cong H_H^*(V) \otimes B$, we can consider quantum multiplication as the map $\star: S \times S \rightarrow S$.

Let $D = q \frac{d}{dq} \in \mathcal{D} = \mathbb{C}[q, q^{-1}, \frac{d}{dq}]$. Consider a (flat) connection ∇ on HQ defined on the sections h^i as

$$\left(\nabla(h^i), q \frac{d}{dq} \right) = k_V \star h^i$$

(the pairing is the natural pairing between differential forms and vector fields). This connection provides the structure of \mathcal{D} -module for S by $D(h^i) = (\nabla(h^i), D)$. Obviously,

$$D \left(\sum_{i=0}^N f_i(q) h^i \right) = \sum_{i=0}^N q \frac{\partial f_i(q)}{\partial q} h^i - h \star \left(\sum_{i=0}^N f_i(q) h^i \right).$$

1.2.4. Definition. The \mathbb{C} -linear operator $D: S \rightarrow S$ is called *the quantum operator*.

Define the operator $D_B: S \rightarrow S$ as $D_B(\sum_{i=0}^N f_i(q) h^i) = \sum_{i=0}^N q \frac{\partial f_i(q)}{\partial q} h^i$. Let $h \star h^j = \sum_j \alpha_{ij} h^i$, $\alpha_{ij} \in B$. Define the matrix M by $M_{ij} = -\alpha_{ij} \in \mathcal{D}$ for $i \neq j$ and $M_{ii} = D - \alpha_{ii} \in \mathcal{D}^3$.

1.2.5. Definition. The differential operator $L_V^Q = \det_{\text{right}}(M) \in \mathcal{D}$ is called *the quantum differential operator* of V .

1.2.6. In the following we study flat solutions of differential equations corresponding to operators we defined. The solutions are “formal series with logarithms” and do not lie in S . So, we need to change the base. Put $T = \mathbb{C}[[q]][t]/(t^{N+1})$. Put $B^{\text{form}} = B \otimes_{\mathbb{C}[q]} T$ and $S^{\text{form}} = S \otimes_{\mathbb{C}[q]} T$. Let \mathcal{D} act on via $Dt = 1$. So, the informal meaning of t is $\log(q)$. In the following we consider D , D_B , $h\star$, and so on as \mathbb{C} -operators $S^{\text{form}} \rightarrow S^{\text{form}}$ and L_V^Q as \mathbb{C} -operator $B^{\text{form}} \rightarrow B^{\text{form}}$.

1.3. Relations. For simplicity we use below Gromov–Witten invariants with negative degree of curve or negative descendants (which are formally not defined). The convention is that they equal 0.

1.3.1. Theorem [Topological recursion, see [Ma99], VI-6.2.1]. Let $\gamma_1, \gamma_2, \gamma_3 \in H^*(V)$, $a_1 \in \mathbb{Z}_{>0}$, $a_2, a_3, d \in \mathbb{Z}_{\geq 0}$. Then

$$\langle \tau_{a_1} \gamma_1, \tau_{a_2} \gamma_2, \tau_{a_3} \gamma_3 \rangle_d = \sum_{d_1+d_2=d, a=0, \dots, N} \langle \tau_{a_1-1} \gamma_1, H^a \rangle_{d_1} \langle H_a, \tau_{a_2} \gamma_2, \tau_{a_3} \gamma_3 \rangle_{d_2}.$$

1.3.2. Theorem [The divisor axiom, see [Ma99], VI-5.4]. Let $\gamma_1, \dots, \gamma_n \in H^*(V)$, $\gamma_0 = rH \in H^2(V, \mathbb{Q})$ be an ample divisor and $a_1, \dots, a_m \in \mathbb{Z}_{\geq 0}$. Then

$$\langle \gamma_0, \tau_{a_1}(\gamma_1), \dots, \tau_{a_m} \gamma_m \rangle_d = rd \langle \tau_{a_1} \gamma_1, \dots, \tau_{a_m} \gamma_m \rangle_d + \sum_{s=1}^m \langle \tau_{a_1} \gamma_1, \dots, \tau_{a_s-1} \gamma_0 \cdot \gamma_s, \dots, \tau_{a_m} \gamma_m \rangle_d.$$

1.3.3. Theorem [The fundamental class axiom, see [Ma99], VI-5.1]. Let $\gamma_1, \dots, \gamma_k \in H^*(V)$, $a_1, \dots, a_k \in \mathbb{Z}_{\geq 0}$. Then

$$\langle \tau_{a_1} \gamma_1, \dots, \tau_{a_k} \gamma_k, H^0 \rangle_d = \sum_{i=1}^k \langle \tau_{a_1} \gamma_1 \dots \tau_{a_{i-1}} \gamma_{i-1}, \tau_{a_i-1} \gamma_i, \tau_{a_{i+1}} \gamma_{i+1}, \dots, \tau_{a_k} \gamma_k \rangle_d.$$

1.4. Fundamental solution. Put $e^{Ht} = \sum_{r=0}^{\infty} \frac{H^r t^r}{r!} \in H_H^*(V) \otimes B^{\text{form}}$ (the sum is finite). Put $\langle \tau_{d_1} t^{\alpha_1} \gamma_1, \dots, \tau_{d_s} t^{\alpha_s} \gamma_s \rangle_d = t^{\sum \alpha_i} \cdot \langle \tau_{d_1} \gamma_1, \dots, \tau_{d_s} \gamma_s \rangle_d$. Consider the matrix Φ with elements

$$\Phi_a^b = \sum_{d \geq 0} q^d \left(\langle \tau_{d+a-b-1} H^b, H_a \rangle_d + \langle \tau_{d+a-b} t H^{b+1}, H_a \rangle_d + \langle \tau_{d+a-b+1} \frac{t^2}{2} H^{b+2}, H_a \rangle_d + \dots \right) = \sum_{d \geq 0} q^d \langle \tau_{\bullet} e^{Ht} H^b, H_a \rangle_d,$$

³Identify any matrix A with entries in \mathcal{D} with operator $S \rightarrow S$ given by $A(\sum f_i h^i) = \sum_i (\sum_j A_{ij} f_j) h^i$. Then M is the matrix of $D = D_B - h\star$.

$0 \leq a, b \leq N$, where the meaning of \bullet in what follows is the number

$$N + d - 3 - \sum_{\text{terms}} (\text{codimension of the cohomological class} - 1).$$

We use the notation $\langle \tau_{\bullet}(H^1 + H^2), H_a \rangle_d$ for $\langle \tau_{d+a-2}H^1, H_a \rangle_d + \langle \tau_{d+a-3}H^2, H_a \rangle_d$ and so on. As two-pointed Gromov–Witten invariants for the degree zero curve are not defined, we put $\langle \tau_{\bullet}e^{H^t}H^b, H_a \rangle_0 = \langle H^0, \tau_{\bullet}e^{H^t}H^b, H_a \rangle_0$.

1.4.1. Proposition [Pandharipande, after Givental, Proposition 2 in [Pa98]]. *Consider the sections $\phi^i = \sum_{a=0}^N \Phi_a^i h^a \in S^{\text{form}}$ (i. e. those that correspond to the columns of Φ^4).*

1) *The sections ϕ^i are flat, i. e. $D\phi^i = 0$.*

2) *If $D\phi = 0$, then $\phi = \sum_{i=0}^N \alpha_i \phi^i$, $\alpha_i \in \mathbb{C}$.*

Proof [Pandharipande]. 1) We need to prove that

$$D_B \phi^i = h \star \phi^i.$$

On the left we have

$$D_B \left(\sum_a \Phi_a^i h^a \right) = \sum_a \sum_{d \geq 0} (dq^d \langle \tau_{\bullet} e^{H^t} H^i, H_a \rangle_{d+1} + q^d \langle \tau_{\bullet} H \cdot e^{H^t} H^i, H_a \rangle_d) h^a = \sum_a \sum_{d \geq 0} q^d \langle \tau_{\bullet} e^{H^t} H^i, H, H_a \rangle_d h^a$$

by the divisor axiom 1.3.2. On the right,

$$h \star \left(\sum_a \Phi_a^i h^a \right) = \sum_s \sum_{d_1, d_2 \geq 0} \sum_a q^{d_1} \langle \tau_{\bullet} e^{H^t} H^i, H_a \rangle_{d_1} q^{d_2} \langle H^a, H, H_s \rangle_{d_2} h^s = \sum_s \sum_d q^{d \geq 0} \langle \tau_{\bullet} e^{H^t} H^i, H, H_s \rangle_d h^s$$

by the topological recursion 1.3.1. Both sides are equal.

2) The constant term of Φ (with respect to t and q) is the identity matrix. This means that the columns of Φ are linearly independent. The differential operator of order $N + 1$ has at most $(N + 1)$ -dimensional space of solutions, so it is generated by $N + 1$ functions ϕ^i . \square

1.4.2. Remark. Consider the matrix M (see Definition 1.2.5). The proposition above states that $M\Phi^i = 0$ for the column-vectors $\Phi^i = (\Phi_0^i, \dots, \Phi_N^i)^T$ that correspond to the sections ϕ^i .

1.4.3. Corollary. Define the matrix Ψ by $\Psi_i^j = \sum_{d \geq 0} q^d \langle \tau_{\bullet} e^{H^t} H_i, H^{N-j} \rangle_d$, $0 \leq i, j \leq N$. Let $\Psi_i = (\Psi_i^0, \dots, \Psi_i^N)^T$ be its column-vectors. Then $M^T \Psi_i = 0$ for $0 \leq i \leq N$.

Proof. Analogous to the proof of Proposition 1.4.1. \square

1.5. The solutions.

1.5.1. Consider any series $I = \sum_{i=0}^N I^i(q) h^i \in \mathbb{C}[[q]][h]/(h^{N+1})$, $I^0(q), \dots, I^N(q) \in \mathbb{C}[[q]]$. Let $I_r = \sum_{i=0}^r \left(I^{r-i}(q) \frac{t^i}{i!} \right) \in T$.

Definition. We say that a series I is the *perturbed solution* of the equation $PI = 0$ (or just the operator $P \in \mathcal{D}$) if $PI_r = 0$ for any $r \leq N$.

In the other words, given $P = P(q, D)$, consider $P_H = P(q, D_B)$ replacing D by D_B . Then I is a perturbed solution of P if and only if $P_H(e^{ht} \cdot I) = 0$.

Recall that the I -series of V is defined by

$$I^V = 1 + \sum_{i,j,d \geq 0} \langle \tau_i H_j \rangle_d h^j q^d \in S^{\text{form}}.$$

1.5.2. Theorem.

1) *The series I^V is the perturbed solution of the equation $L_V^Q I = 0$.*

2) *If $L_V^Q I = 0$, then $I = \sum a_i I_i^V$ for some $a_0, \dots, a_N \in \mathbb{C}$.*

⁴ Informally, Φ is “the matrix of fundamental solutions of equation given by the quantum operator in the standard basis”.

Proof. 1) We have

$$L_V^Q I_i^V = L_V^Q \left(\frac{t^i}{i!} + \sum_{d \geq 0} q^d \langle \tau_\bullet e^{Ht} H_i \rangle_d \right) = L_V^Q \left(\sum_{d \geq 0} q^d \langle \tau_\bullet e^{Ht} H_i, H^0 \rangle_d \right) =$$

$$\det_{\text{right}}(M) \Psi_i^N = \det_{\text{right}}(M^{\tau\tau}) \Psi_i^N = 0$$

by Lemma 1.1.4 and Corollary 1.4.3.

2) The solutions I_s^V are linearly independent (by Proposition 1.4.1), so they form a basis of $(N+1)$ -dimensional space of solutions of differential equation of order $N+1$ associated with L_V^Q . \square

2. DN's

2.1. Let

$$L_V^Q = P_{V,0}(D) + qP_{V,1}(D) + \dots + q^n P_{V,n}(D) \in \mathcal{D}$$

be a quantum differential operator of a quantum minimal smooth Fano variety of dimension N (usually $n = N+1$). Its singularities are not regular in general.

2.1.1. Definition [see [Go05], 1.9]. The operator

$$\tilde{L}_V = P_{V,0}(D) + qP_{V,1}(D) \cdot (D+1) + \dots + q^n P_{V,n}(D) \cdot (D+1) \cdot \dots \cdot (D+n)$$

is called *the regularization* of L_V^Q .

The singularities of all known \tilde{L}_V are regular⁵. Obviously, \tilde{L}_V is divisible by D on the left.

2.1.2. Definition [see [Go05], Definition 2.10]. The operator L_V such that $DL_V = \tilde{L}_V$ is called *the (geometric) operator of type DN*.

The solutions of equations associated with geometric operators of type DN are conjectured to be G -series⁶.

Consider any differential operator

$$P = P_0(D) + qP_1(D) + \dots + q^n P_n(D) \in \mathbb{C}[q, q \frac{d}{dq}].$$

Let

$$\tilde{P} = P_0(D) + qP_1(D) \cdot (D+1) + \dots + q^n P_n(D) \cdot (D+1) \cdot \dots \cdot (D+n)$$

be a regularization as before.

2.2. The Frobenius method. We describe an “algebraic” interpretation of the Frobenius method of solving differential equations. For the standard version see [CL55], IV–8.

Let $R = \mathbb{C}[\varepsilon]/(\varepsilon^{N+1})$, $N+1 \in \mathbb{N}$. Consider the differential operator

$$P_\varepsilon = P_0(D + \varepsilon) + qP_1(D + \varepsilon) + \dots + q^n P_n(D + \varepsilon) \in \mathcal{D} \otimes R.$$

2.2.1. Definition. Consider the sequence $\{\bar{c}_i\}$, $i \geq 0$, $\bar{c}_i \in R$. It is called *a Newton solution* of P_ε , if for any $m \in \mathbb{Z}$

$$\bar{c}_m P_n(m + \varepsilon) + \bar{c}_{m+1} P_{n-1}(m + 1 + \varepsilon) + \dots + \bar{c}_{m+n} P_0(m + n + \varepsilon) = 0$$

(the convention is that \bar{c}_i 's with negative subscripts are 0).

2.2.2. Proposition. *The sequence $\{\bar{c}_i\}$ is a Newton solution of P_ε if and only if the series*

$$I = \bar{c}_0 + q\bar{c}_1 + \dots \in \mathbb{C}[[q]] \otimes R$$

is a perturbed solution of P .

Proof. Recall that $T = \mathbb{C}[[q]][t]/(t^{N+1})$. We consider T in the proof as a \mathbb{C} -vector space. Consider the linear space (over \mathbb{C})

$$C = \{(\bar{a}_0, \bar{a}_1, \dots), \bar{a}_i \in R\}$$

⁵They are also regular if the matrix of quantum multiplication by the anticanonical class is diagonalizable, see [GS05], Remark 3.6.

⁶That is, for any solution of type $I = \sum a_i q^i$, $a_n \in \mathbb{Q}$, the following conditions hold. Let $a_n = \frac{p_n}{q_n}$, $(p_n, q_n) = 1$, $q_n \geq 1$. Then I has positive radii of convergence in \mathbb{C} and \mathbb{Q}_p for any prime p and there exist a constant $C < \infty$ such that $\text{LCM}(q_1, q_2, \dots, q_n) < C^n$ for any n .

with basis $\{b_{ij} = (0, \dots, 0, \varepsilon^{N-j}, 0, \dots), i \geq 0, 0 \leq j \leq N\}$ (ε^{N-j} is in the i -th place) and the isomorphism $l: C \rightarrow T$ given by $b_{ij} \mapsto q^{it^j}/j!$. It is easy to see that the formulas $q \cdot b_{ij} = b_{i+1,j}$ and $D \cdot b_{ij} = (i + \varepsilon)b_{ij}$ determine the action of \mathcal{D} on C . Trivially, $l(q \cdot b_{ij}) = q \cdot l(b_{ij})$ and $l(D \cdot b_{ij}) = D \cdot l(b_{ij})$, i. e. the actions of \mathcal{D} on C and T commute. Thus, $P(\{\tilde{c}_i\}) = 0$ if and only if $P(I_N) = 0$ (we follow the notations of 1.5.1 with $h = \varepsilon$). Analogously, if $P(\{\tilde{c}_i\}) = 0$, then $P(I_r) = 0$ for $0 \leq r \leq N$. \square

2.2.3. Remark. A Newton solution of P_ε exists in R if and only if $\text{mult}_0 P_0 \geq N + 1$.

2.2.4. Remark. We consider the case $P_0(0) = 0$. The cases of the other roots of P_0 are of this type after shifting of variables.

2.2.5. Corollary. Let $I = \sum a_{ij} q^i \varepsilon^j \in \mathbb{C}[[q]] \otimes R$ be a perturbed solution of P . Then

$$\tilde{I} = \sum a_{ij} q^i \varepsilon^j \cdot (\varepsilon + 1) \cdot \dots \cdot (\varepsilon + i)$$

is the perturbed solution of \tilde{P} .

Proof. It follows from the formula for \tilde{P} and Proposition 2.2.2. \square

2.2.6. Corollary. Let V be a smooth quantum minimal Fano variety of dimension N and L_V^Q be the corresponding quantum differential operator. Let L_V be the corresponding operator of type DN . Define the polynomials $I^i(h)$ in h by $I^V = \sum_{i=0}^{\infty} I^i(h) q^i$.

1) Let

$$\tilde{I}^V = \sum_{i=0}^{\infty} I^i(h) \cdot (h + 1) \cdot \dots \cdot (h + i) q^i.$$

Then $\tilde{I}^V \bmod h^N$ is the perturbed solution of $L_V I = 0$.

2) If $L_V I = 0$, then $I = \sum a_i \tilde{I}_i^V$ for some $a_0, \dots, a_{N-1} \in \mathbb{C}$.

Proof. 1) By Theorem 1.5.2 I^V is the perturbed solution of L_V^Q . Let \tilde{L}_V be the regularization of L_V^Q . Then, by Corollary 2.2.5, \tilde{I}^V is a perturbed solution of \tilde{L}_V . The relations for the Newton solution for $L_{V,\varepsilon}$ are proportional to the corresponding relations for $\tilde{L}_{V,\varepsilon}$ modulo ε^N (we identify the parameter ε in the Frobenius method with h). So, the Newton solutions of $\tilde{L}_{V,\varepsilon}$ and $L_{V,\varepsilon}$ coincide modulo ε^N .

2) It follows from standard arguments on linear independence (see the proof of theorem 1.5.2). \square

2.3. Example. The matrix of quantum multiplication for \mathbb{P}^N is

$$\begin{pmatrix} 0 & 0 & \dots & 0 & (N+1)^{N+1} q^{N+1} \\ 1 & 0 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

The corresponding quantum differential operator is

$$L_{\mathbb{P}^N}^Q = D^{N+1} - (N+1)^{N+1} q^{N+1}.$$

Let F be a class dual to the hyperplane in \mathbb{P}^N (so, $-K_{\mathbb{P}^N} = (N+1)F$) and $f = F \otimes 1 \in S^{form}$. It is easy to see that the series

$$I^{\mathbb{P}^N} = \sum_{d \geq 0} \frac{q^{(N+1)d}}{(f+1)^{N+1} \cdot \dots \cdot (f+d)^{N+1}}$$

is a perturbed solution of $L_{\mathbb{P}^N}^Q \Phi = 0$.

The operator of type DN for \mathbb{P}^N is

$$L_{\mathbb{P}^N} = D^N - (N+1)^{N+1} q^{N+1} (D+1) \cdot \dots \cdot (D+N)$$

and a perturbed solution of this operator is the series

$$\tilde{I}^{\mathbb{P}^N} = \sum_{d \geq 0} \frac{q^{(N+1)d} (h+1) \cdot \dots \cdot (h+(N+1)d)}{(f+1)^{N+1} \cdot \dots \cdot (f+d)^{N+1}}.$$

3. UNIVERSALITY OF DN 'S AND SOLUTIONS

All formulas above are formal consequences of the formulas 1.3.1–1.3.3. So, the natural idea is to define “abstract Gromov–Witten theory”, that is, to consider Gromov–Witten invariants as formal variables with natural relations. Moreover, if we consider “invariants” that correspond to several classes of type H^i and one Poincaré dual class of type H_r we may develop a universal abstract Gromov–Witten theory that do not depend on the dimension N .

A convenient multiplicative basis of GW is given by the First Reconstruction Theorem from [KM94]. We follow the notations of the definition of GW on page 2.

3.1. Theorem [The Abstract Reconstruction Theorem]. *The map $r: \mathbb{C}[a_{ij}] \rightarrow GW$ is an isomorphism.*

Proof. The following relation is the formal implication from the relations of type GW5.

GW6: For any finite subset $S \subset \mathbb{N}$ denote $H^{i_{s_4}}, \dots, H^{i_{s_k}}$ (where s_j 's are distinct elements of S) by \coprod_S . Then for any $n \geq 0$

$$\sum_{S_1} \langle \coprod_{S_1}, H^{i_1}, H^{i_2}, H_a \rangle \langle H^a, \coprod_{S_2}, H^{i_3}, H_r \rangle = \sum_{T_1} \langle \coprod_{T_1}, H^{i_1}, H^{i_3}, H_b \rangle \langle H^b, \coprod_{T_2}, H^{i_2}, H_r \rangle,$$

where the sums are taken over all splittings $S_1 \sqcup S_2 = \{4, \dots, n\}$, $T_1 \sqcup T_2 = \{4, \dots, n\}$ and all a and b such that the degrees of all symbols are non-negative (notice that both sums are finite).

These relations are called quadratic relations in the geometrical case (see. [KM94], 3.2.2). If S is empty, then these relations are called associativity equations or WDVV equations. Though these relations follows from GW5, we include them in the generators of relations ideal of GW for simplicity.

Let us prove that r is epimorphic. Our proof is an abstract version of one in [KM94]. We denote $r(a_{ij})$ also by a_{ij} for simplicity. We need to prove that any “invariant” $\langle \tau_{d_1} H^{i_1}, \dots, \tau_{d_n} H_r \rangle$ can be expressed in terms of a_{ij} 's.

Applying relation GW4 to one- or two-pointed invariants (the abstract symbols), we may assume $n \geq 3$ (see the proof of Proposition 5.2 in [Pr04]). Using GW5 (and GW2), we may assume that all d_i 's equal 0.

Given an invariant $C = \langle H^{i_1+1}, \dots, H^{i_n}, H_r \rangle$, $n \geq 2$, $i_1 > 1$, write GW6 for classes $H, H^{i_1}, \dots, H^{i_n}, H_r$. We see that, modulo invariants with lower number of terms, GW4 and GW2, C equals the sum of the invariants with terms $H^{i_1}, \dots, H^{i_n}, H_r$. So, using GW1–GW6 we may express any invariant in terms of three-pointed invariants with H^1 as the first term, that is, in terms of a_{ij} 's. Thus, r is an epimorphism.

Let us prove that r is a monomorphism step by step.

Step 1. Let GW_{3p} and GW_{4p} be the relations of type GW3 and GW4 for invariants without descendants. Then the ideal Rel is generated by GW1, GW2, GW_{3p} , GW_{4p} , GW5, GW6 (the relations “commute”). Notice that GW_{3p} is a particular case of GW2.

Step 2. Let

$$GW' = \mathbb{C}[F'] / (GW_{4p}, GW5, GW6),$$

where $F' \subset F$ are invariants of positive degree of type

$$\langle \tau_{d_1} H^{i_1}, \dots, \tau_{d_{n-1}} H^{i_{n-1}}, \tau_{d_n} H_r \rangle,$$

with $i_k \geq i_l$ for $k > l$ and $d_k \geq d_l$ if $i_k = i_l$. (Thus, the left side of any relation of type GW2 becomes just the notation of the number on the right side.) Obviously, $GW' \cong GW$.

Step 3. Let $A_p = \mathbb{C}[F'_p] / (GW_{4p}, GW6)$, where $F'_p \subset F'$ is the subset of invariants without descendants. Let us prove that the natural map $A_p \rightarrow GW'$ is a monomorphism. Consider the order on the invariants, that is, the function w on F'_p given by

$$w(\langle \tau_{d_1} H^{i_1}, \dots, \tau_{d_{n-1}} H^{i_{n-1}}, \tau_{d_n} H_r \rangle) = (\sum d_j, d_1, i_1, \dots, d_n, r).$$

We say that $C_1 > C_2$ if $w(C_1) > w(C_2)$ (with respect to the natural lexicographic order). Define the lexicographic order on the monomials in F' , that is, for any two monomials $M_1 = \alpha \cdot C_1^{a_1} \cdot \dots \cdot C_n^{a_n}$, $M_2 = \beta \cdot C_1^{b_1} \cdot \dots \cdot C_n^{b_n}$ (where $\alpha, \beta \in \mathbb{C}$ and $C_1 > C_2 > \dots > C_n$) say that $M_1 > M_2$ if $a_1 > b_1$, or $a_1 = b_1$ and $a_2 > b_2$, and so on. Denote the leading term of $E \in \mathbb{C}[F']$ with respect to this order by $L(E)$. Denote the relation of type GW5 with the invariant C on the left

side (which is not uniquely defined!) by $\text{GW5}(C)$. For any prime (i. e. without descendants) invariant C put $\text{GW5}(C) = C$. Consider any $P \neq 0$ in $\mathbb{C}[F'_p]$ such that $r(P) \in (\text{GW4}_p, \text{GW5}, \text{GW6}) \triangleleft \mathbb{C}[F']$ for the natural map $r: \mathbb{C}[F'_p] \rightarrow \mathbb{C}[F']$. Denote $r(P)$ by P for simplicity. Let $P = \sum_{j \in J} \beta_j \cdot \prod_{i \in I} C_i^{b_{i,j}} \text{GW5}(C_j)$ modulo $(\text{GW4}_p, \text{GW6})$, where $\beta_j \in \mathbb{C}$. Applying GW4 we may assume that invariants C_i 's contains at least three terms. That is, we can apply relation of type GW5 to them. Denote the maximal of the leading terms of all summands of type $\prod_{i \in I} C_i^{b_{i,j}} \text{GW5}(C_j)$ by L . Let $J_0 \subset J$ be the subset of indices such that $L(\prod_{i \in I} C_i^{b_{i,j}} \text{GW5}(C_j)) = L$ for $j \in J_0$. Let L has a factor with descendants. Then we have

$$P = \sum_{j \in J_0} \beta_j \cdot \prod_{i \in I} C_i^{b_{i,j}} \text{GW5}(C_j) + (\text{summands with smaller leading terms}).$$

Obviously, $L(\text{GW5}(C)) = C$. One may check that the difference of two relations of type $\text{GW5}(C)$ may be expressed in terms of relations of type GW5 with smaller leading terms and relations of type GW6. Thus, expressions of type $\text{GW5}(C_i)$ in the sum on the right side coincides for every i modulo summands with smaller leading terms. Then

$$P = \sum_{j \in J_0} \beta_j \cdot \prod_{i \in I} \text{GW5}(C_i)^{b_{i,j}} \text{GW5}(C_j) + (\text{summands with smaller leading terms}) = \sum_{j \in J_0} \beta_j \cdot \left(\prod_{i \in I \cup J_0} \text{GW5}(C_i)^{c_i} \right) + (\text{summands with smaller leading terms}).$$

As $L(P) < L$, we have $\sum_{j \in J_0} \beta_j = 0$. We get the expression for P with smaller L . Repeating this procedure, we obtain the expression for P with $L(P) = L$, i. e. without relations of type GW5. Thus, $P \in (\text{GW4}_p, \text{GW6})$ and $A_p \cong \text{GW}' \cong \text{GW}$, i. e. invariant may be uniquely expressed in terms of prime ones.

Step 4. Let us prove that any prime invariant may be uniquely expressed in terms of a_{ij} 's. We call the invariants of type $\langle H^k, H^1, \dots, H^1, H_r \rangle$ trivial since the relations of type GW6 for them are trivial. Let $F_t \subset F'_p$ be the subset of trivial invariants. Obviously, $A \cong A_t = \mathbb{C}[F_t]/(\text{GW4}_p) \cong \mathbb{C}[F_t]/(\text{GW4}_p, \text{GW6})$. Let $F'_t \subset F'_p$ be the subset of invariants without terms H^1 and $\text{GW6}'$ be the relations of type GW6 with invariants with terms H^1 replaced by ones without such terms given by GW4_p . Let us prove that $A_t \cong \mathbb{C}[F'_t]/(\text{GW6}') \cong A_p$.

Define the function w' on the elements of F'_t given by

$$w'(\langle H^{i_1}, \dots, H^{i_{n-1}}, H_r \rangle) = (n, i_1, \dots, i_{n-1}, r).$$

Define the order on monomials in F' and the leading term $L'(E)$ of any $E \in \mathbb{C}[F'_p]$ as before. The direct computation shows that the difference of the two relations of type $\text{GW6}'$ with the same leading terms may be expressed in terms of the relations of type $\text{GW6}'$ with the smaller leading terms ("the relations of type GW6 commute"). Assume that $P = P(a_{ij}) \in (\text{GW6}') \triangleleft \mathbb{C}[F'_t]$. As before, we may obtain the expression for P in terms of relations of type $\text{GW6}'$ containing only trivial invariants. Since these relations vanish, $P = 0$ and $A \cong A_p \cong \text{GW}$. \square

3.2. Remark. So, the Gromov–Witten theory of a quantum minimal Fano variety V of dimension N is a particular function from $\text{GW}_N = r(i_N(A_N))$ to \mathbb{C} , where $i_N: A_N \rightarrow \text{GW}$ is given by $i_N(a_{ij}) = a_{ij}$ if $(i, j) \neq (0, 0)$ and $i_N(a_{00}) = 0$.

Theorem 3.1 enables us to define the universal I -series $I \in A \otimes \mathbb{C}[[q]][[h]]$ such that for any N the abstract I -series for dimension N

$$I^N = \sum_{i,j} \langle \tau_i H_j \rangle \cdot q^d h^j \in \text{GW}_N \otimes \mathbb{C}[[q]][h]/h^{N+1}$$

is the restriction of I , that is, $I^N = r_N(I \bmod h^{N+1})$. Analogously, we may define the universal "regularized I -series" \tilde{I} such that for

$$\tilde{I}^N = \sum_{i,j} \langle \tau_i H_j \rangle \cdot q^d h^j \cdot (h+1) \cdot \dots \cdot (h+d) \in \text{GW}_N \otimes \mathbb{C}[[q]][h]/h^{N+1}$$

we have $\tilde{I}^N = r_N(\tilde{I} \bmod h^{N+1})$.

Consider the torus $\mathbb{T} = \text{Spec } \mathbb{C}[q, q^{-1}]$ and the trivial vector bundle HQ^N with fiber $GW_N \otimes \langle H^0, H^1, \dots, H^N \rangle$ (H^i 's are just notations for basis vectors). Let $h^i = 1 \otimes H^i$. Let

$$A^N = \begin{pmatrix} a_{0,0}q & a_{0,1}q^2 & \dots & a_{0,N-1}q^N & a_{0,N}q^{N+1} \\ 1 & a_{1,1}q & \dots & a_{1,N-1}q^{N-1} & a_{1,N}q^N \\ & & \dots & & \\ 0 & 0 & \dots & 1 & a_{N,N}q \end{pmatrix}$$

(where $a_{00} = 0$). Define the abstract quantum connection ∇^N by

$$\left(\nabla^N(h^i), q \frac{d}{dq} \right) = A^N h^i$$

(the connection commutes with a_{ij}). Repeat all the previous for the abstract case. In particular, define the abstract quantum differential operator $L_N^Q \in GW_N \otimes \mathcal{D}$ and the operator $L_N \in GW_N \otimes \mathcal{D}$ (recall that after specialization of abstract Gromov–Witten invariants to the geometric ones this operator is called geometric DN). (It is easy to see that this operator is the same as one defined on the page 2.) Then we obtain the following theorem.

3.3. Theorem.

- 1) The series I^N is the perturbed solution of $L_N^Q I = 0$.
- 2) The series $\tilde{I}^N \bmod h^N$ is the perturbed solution of $L_N I = 0$.

In the other words, given \tilde{I}^N (resp. L_N), one should put $a_{ij} = 0$ for $N_0 < i, j \leq N$ to obtain \tilde{I}^{N_0} (resp. $D^{N-N_0} L_{N_0}$).

3.4. Remark. The same holds for DN 's. Recall that operator of type DN is L_N with identified a_{ij} and $a_{N-j, N-i}$. Let J^N be a perturbed solution of DN . If $n \ll N$ and $N < N_0$, then

$$J^N \bmod (q^n) = J^{N_0} \bmod (q^n, h^N).$$

3.5. Remark. Define L_N^Q and L_N as operators in $\mathbb{C}[a_{ij}]$, $0 \leq i \leq j$ (i. e. let a_{00} be non-zero). Then the universality for their solutions also holds. The universal series for L_N^Q is $e^{a_{00}q} \cdot I'$, where I' is given from I by shift $a_{ii} \mapsto a_{ii} - a_{00}$, and the universal series for L_N is the regularization of $e^{a_{00}q} \cdot I'$.

4. APPENDIX

Consider a differential operator $P = \sum_{i=0}^N q^i P_i(D) \in \mathcal{D}$. Denote the r -th formal derivative of P with respect to D by $P^{(r)}$.

4.1. Theorem. The series $I = \sum_{i=0}^N I^i h^i$ (I^i 's are series in q) is a perturbed solution of P if and only if for any $s \leq N$

$$\frac{P^{(s)}(I^0)}{s!} + \frac{P^{(s-1)}(I^1)}{(s-1)!} + \dots + P(I^s) = 0.$$

Proof. Notice that

$$P(tJ(q)) = tP(J(q)) + P^{(1)}(J(q))$$

for any $J(q) \in \mathbb{C}[[q]]$ (see [BvS95], Proposition 4.3.1). Thus,

$$P(t^r J) = \sum_{i=0}^r \binom{i}{r} t^i P^{(r-i)}(J).$$

For any $s \leq N$

$$\begin{aligned} P(I_s) &= P \left(\frac{t^s}{s!} I^0 + \frac{t^{s-1}}{(s-1)!} I^1 + \dots + I^s \right) = \sum_{\alpha=0}^s P \left(\frac{t^\alpha}{\alpha!} I^{s-\alpha} \right) = \\ &= \sum_{\alpha=0}^s \sum_{\beta=0}^\alpha \left(\binom{\beta}{\alpha} \cdot \frac{t^\beta P^{(\alpha-\beta)}(I^{s-\alpha})}{\alpha!} \right) = \sum_{\alpha=0}^s \sum_{\beta=0}^\alpha \left(\frac{t^\beta P^{(\alpha-\beta)}(I^{s-\alpha})}{\beta! (\alpha-\beta)!} \right) = \sum_{\alpha=0}^s \sum_{\beta=0}^\alpha R_{\alpha,\beta}, \end{aligned}$$

where

$$R_{\alpha,\beta} = \frac{t^\beta P^{(\alpha-\beta)}(I^{s-\alpha})}{\beta! (\alpha-\beta)!}.$$

Prove the theorem by induction on s . Suppose that it holds for any $s_0 < s$. Then

$$\begin{aligned} \frac{P^{(s)}(I^0)}{s!} + \frac{P^{(s-1)}(I^1)}{(s-1)!} + \dots + P(I^s) &= \sum_{a=0}^s \frac{t^a}{a!} \left(\frac{P^{(s-a)}(I^0)}{(s-a)!} + \dots + P(I^{s-a}) \right) = \\ &= \sum_{a=0}^s \sum_{b=0}^{s-a} \frac{t^a P^{(b)}(I^{s-a-b})}{a!b!} = \sum_{a=0}^s \sum_{b=0}^{s-a} S_{a,b}, \end{aligned}$$

where

$$S_{a,b} = \frac{t^a P^{(b)}(I^{s-a-b})}{a!b!}.$$

Obviously, $S_{a,b} = R_{a+b,a}$, so

$$\sum_{a=0}^s \sum_{b=0}^{s-a} S_{a,b} = \sum_{a=0}^s \sum_{b=0}^{s-a} R_{a+b,a} = \sum_{\alpha=0}^s \sum_{\beta=\alpha}^s R_{\beta,\alpha} = \sum_{0 \leq \alpha \leq \beta \leq s} R_{\beta,\alpha} = \sum_{a=0}^s \sum_{b=0}^a R_{a,b}.$$

Thus,

$$\frac{P^{(s)}(I^0)}{s!} + \frac{P^{(s-1)}(I^1)}{(s-1)!} + \dots + P(I^s) = P(I_s),$$

which proves the theorem. \square

4.2. Remark. For $s = 1$ this theorem is proven in [BvS95], Proposition 4.3.2 and for $s \leq 2$ in [Tj98], Appendix B.

4.3. Proposition [Newton method]. *The series*

$$\Phi = a_0 + a_1 q + a_2 q^2 + \dots \in B, \quad a_i \in \mathbb{C}.$$

is a solution of $P\Phi = 0$ (as a formal series) if and only if for any $m \in \mathbb{Z}$

$$a_m P_N(m) + a_{m+1} P_{N-1}(m+1) + \dots + a_{m+N} P_0(m+N) = 0,$$

where a_i 's with negative subscripts are assumed to be 0.

Proof. Straightforward. \square

Theorem 4.1 and Proposition 4.3 enable us to find the relations for the solutions of $P\Phi = 0$.

4.4. Corollary. Let $I = \sum a_{ij} h^i q^j$. Then I is a perturbed solution of P if and only if for any $s \leq N$ and for any $m \in \mathbb{N}$

$$\begin{aligned} &\frac{a_{0,m} P_N^{(s)}(m) + a_{0,m+1} P_{N-1}^{(s)}(m+1) + \dots + a_{0,m+N} P_0^{(s)}(m+N)}{s!} + \\ &\frac{a_{1,m} P_N^{(s-1)}(m) + a_{1,m+1} P_{N-1}^{(s-1)}(m+1) + \dots + a_{1,m+N} P_0^{(s-1)}(m+N)}{(s-1)!} + \dots + \\ &a_{s,m} P_N(m) + a_{s,m+1} P_{N-1}(m+1) + \dots + a_{s,m+N} P_0(m+N) = 0. \end{aligned}$$

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